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Well-posedness for the Cauchy problem of coupled Hirota equations with low regularity data ☆

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Abstract

The local well-posedness of the Cauchy problem for the coupled Hirota equations is established in Sobolev spaces $H^s \times H^s$ ($s \geqslant \frac{1}{4}$) by the Fourier restriction norm method.

Keywords: Coupled Hirota equations; Fourier restriction norm; Cauchy problem; Trilinear estimates

1. Introduction

We study Cauchy problem for the coupled Hirota equations:

$$
\begin{aligned}
&i\partial_t u + c_1\partial_x^2 u - i\varepsilon\partial_x^3 u + 2\big(\alpha|u|^2 + \beta|v|^2\big)u - i\varepsilon\big\{\big(2\mu_1|u|^2 + \nu_1|v|^2\big)u_x + \nu_1 u\bar{v}v_x\big\} = 0,\\
&i\partial_t v + c_2\partial_x^2 v - i\varepsilon\partial_x^3 v + 2\big(\beta|u|^2 + \gamma|v|^2\big)v - i\varepsilon\big\{\big(\nu_2|u|^2 + 2\mu_2|v|^2\big)v_x + \nu_2\bar{u}vu_x\big\} = 0,\\
&u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \qquad x, t \in \mathbb{R},
\end{aligned}
\tag{1.1}
$$

where $(u_0(x), v_0(x)) \in H^s \times H^s$, the coefficients c_j $(j = 1, 2)$ and $\varepsilon \neq 0$ are real constants and α, β, γ, μ_j, ν_j $(j = 1, 2)$ are complex constants, the unknowns u and v are complex valued functions, and $\bar{u}$, $\bar{v}$ denote their conjugate functions.

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The coupled Hirota equations describe pulse propagation in a coupled fiber with higher-order dispersion and self-steepening. Several papers have been devoted to the study of soliton solutions, see [4,5]. So far there is no result on the Cauchy problem for the coupled Hirota equations. Here we are interested in the well-posedness for the Cauchy problem of the coupled Hirota equations with low regularity data. The classical theory of PDE focuses on smooth solutions, when s is extremely large for space H^s. In this case it is fairly easy to obtain local well-posedness for the coupled Hirota equations, but it is sometimes necessary to go down to much lower regularity. The low regularity theory is also useful in obtaining more precise control on solutions (including smooth solutions) and more generally in deepening the intuition and set of tools for PDEs.

The single Hirota equation encompasses the well-known nonlinear derivative Schrödinger equation

$$iu_t + u_{xx} + i\left(|u|^2 u\right)_x = 0 \tag{1.2}$$

and the modified KdV equation (m-KdV)

$$u_t + u_{xxx} + u^2 u_x = 0. \tag{1.3}$$

For nonlinear Schrödinger equations with derivative in a nonlinear term, H. Takaoka used the trilinear estimates associated with the Fourier restriction norm method to obtain the global well-posedness with data in low-order Sobolev spaces [11], Colliander et al. [2,3] established sharp results on global well-posedness for the initial value problem. The Fourier restriction norm method [1,6,8,11] was first introduced by Bourgain [1] to study the KdV and nonlinear Schrödinger equations in the periodic case. It was simplified by Kenig et al. in dealing with KdV equation [6,8]. Kenig et al. [9] proved that the Cauchy problem of m-KdV equation (1.3) is locally well-posed in H^s with the best order $s \geqslant \frac{1}{4}$.

In this paper, we will prove that Cauchy problem (1.1) is locally well-posed in $H^s \times H^s$ with $s \geqslant \frac{1}{4}$ by the Fourier restriction norm method and the contraction mapping principle. The most important step is to prove the trilinear estimates with the dispersion relation attuned to the coupled Hirota equations. Compared with results of [9], the single Hirota equation and the KdV equation have the same highest order dispersive term iu_{xxx}, our result with $s \geqslant \frac{1}{4}$ for the Cauchy problem (1.1) seems difficult to be improved.

In order to study the Cauchy problem, we use the equivalent formulation

$$\begin{aligned} u(x,t) &= S_1(t)u_0 \\ &\quad - i\int_0^t S_1(t-t')\left[2\left(\alpha|u|^2+\beta|v|^2\right)u - i\varepsilon\left\{\left(2\mu_1|u|^2+\nu_1|v|^2\right)u_x + \nu_1 u\bar{v}v_x\right\}\right]dt', \\ v(x,t) &= S_2(t)v_0 \\ &\quad - i\int_0^t S_2(t-t')\left[2\left(\beta|u|^2+\gamma|v|^2\right)v - i\varepsilon\left\{\left(\nu_2|u|^2+2\mu_2|v|^2\right)v_x + \nu_2\bar{u}vu_x\right\}\right]dt', \end{aligned}$$

where

$$S_1(t) = \mathcal{F}_x^{-1}e^{-it(c_1\xi^2+\varepsilon\xi^3)}\mathcal{F}_x, \qquad S_2(t) = \mathcal{F}_x^{-1}e^{-it(c_2\xi^2+\varepsilon\xi^3)}\mathcal{F}_x$$

are the unitary operators associated to the corresponding linear equations. We denote the phase function by

$$\phi_j(\xi) = \varepsilon\xi^3 + c_j\xi^2, \quad j = 1, 2. \tag{1.4}$$

It is important to point out that the phase functions $\phi_j(\xi)$ $(j=1,2)$ above or their derivatives have nonzero singular points. This is different from the phase function of the semigroup of the linear KdV equation and also makes the problem much more difficult. Therefore, we need to use Fourier restriction operators

$$P^N f = \int\limits_{|\xi|\geqslant N} e^{ix\xi}\hat f(\xi)\,d\xi, \quad P_N f = \int\limits_{|\xi|\leqslant N} e^{ix\xi}\hat f(\xi)\,d\xi, \quad \forall N>0 \tag{1.5}$$

to eliminate the singularity of the phase function.

Moreover, the operators will be used to decompose the coupled terms and nonlinear terms in the system (1.1). For instance, to deal with terms like $(u_1)_x u_2 \bar u_3$ $(u_1,u_2,u_3 = u$ or $v)$, we first decompose it as the high frequency part and corresponding low one

$$(u_1)_x u_2\bar u_3 = P^N\big\{(u_1)_x u_2\bar u_3\big\} + P_N\big\{(u_1)_x u_2\bar u_3\big\}. \tag{1.6}$$

Next, we continue to decompose each term in the right side of (1.6) as the sum of those products which consist of each factor acted by the Fourier restriction operator P^N or P_N. We will estimate each resulting term with different methods to overcome any obstacles. In order to give our main result, we shall introduce some definitions and notations first.

Definition 1.1. For $s,b\in\mathbb{R}$, we define the spaces $X^j_{s,b}$ and $\bar X^j_{s,b}$ for the coupled Hirota equations to be the completion of the Schwartz function space on $\mathbb{R}^2$ with respect to the norms

$$\|u\|_{X^j_{s,b}} = \big\|S_j(-t)u\big\|_{H^b_t H^s_x} = \big\|\langle\xi\rangle^s\big\langle\tau-\varepsilon\xi^3-c_j\xi^2\big\rangle^b\mathcal{F}u\big\|_{L^2_\xi L^2_\tau}$$

and

$$\|\bar u\|_{\bar X^j_{s,b}} = \big\|\langle\xi\rangle^s\big\langle\tau-\varepsilon\xi^3+c_j\xi^2\big\rangle^b\mathcal{F}\bar u\big\|_{L^2_\xi L^2_\tau},$$

respectively, where $j=1,2$ and $\langle\cdot\rangle = (1+|\cdot|)$.

One can easily prove that $\|u\|_{X^j_{s,b}} = \|\bar u\|_{\bar X^j_{s,b}}$, which will be used later without pointing out. We shall use the trivial embedding: for $j=1,2$, it holds that $\|u\|_{X^j_{s_1,b_1}} \leqslant \|u\|_{X^j_{s_2,b_2}}$ whenever $s_1\leqslant s_2$, $b_1\leqslant b_2$. We denote by $\hat u = \mathcal{F}u$ the Fourier transform in t and x of u and by $\mathcal{F}_{(\cdot)}u$ the Fourier transform in the $(\cdot)$ variable, respectively. Let us introduce the notations, for $j=1,2$, $k=1,2,3$,

$$\begin{aligned}
&\sigma^j = \tau-\varepsilon\xi^3-c_j\xi^2, \qquad \sigma^j_1 = \tau_1-\varepsilon\xi_1^3-c_j\xi_1^2,\\
&\sigma^j_2 = \tau_2-\varepsilon\xi_2^3-c_j\xi_2^2, \qquad \sigma^j_3 = \tau_3-\varepsilon\xi_3^3+c_j\xi_3^2,\\
&\sigma = \sigma^1 \text{ or } \sigma^2, \qquad \sigma_k = \sigma^1_k \text{ or } \sigma^2_k,\\
&\phi = \phi_1 \text{ or } \phi_2, \qquad S(t) = S_1(t) \text{ or } S_2(t), \qquad X_{s,b} = X^1_{s,b} \text{ or } X^2_{s,b}.
\end{aligned} \tag{1.7}$$

We use the notation $\int_\star \cdot\, d\delta$ to simplify the convolution integral:

$$\begin{aligned}
&\int\limits_\star f\big(\{\xi,\tau\},\{\xi_1,\tau_1\},\{\xi_2,\tau_2\},\{\xi_3,\tau_3\}\big)\,d\delta\\
&\quad = \int f\big(\{\xi_1+\xi_2+\xi_3,\tau_1+\tau_2+\tau_3\},\{\xi_1,\tau_1\},\{\xi_2,\tau_2\},\{\xi_3,\tau_3\}\big)\,d\tau_1\,d\tau_2\,d\tau_3\,d\xi_1\,d\xi_2\,d\xi_3.
\end{aligned}$$

Let $\psi \in C_0^\infty(\mathbb{R})$ with $\psi = 1$ on $[-1/2, 1/2]$ and $\operatorname{supp}\psi \subset [-1, 1]$. Denote $\psi_\delta(\cdot) = \psi(\delta^{-1}(\cdot))$ for any nonzero $\delta \in \mathbb{R}$.

Theorem 1.1. *Let $s \geqslant \frac{1}{4}$, $\frac{1}{2} < b < \frac{2}{3}$. Then there exists a constant $T > 0$ such that the Cauchy problem* (1.1) *admits a unique local solution*

$$\big(u(x,t), v(x,t)\big) \in \big\{C\big([0,T]; H^s\big) \cap X^1_{s,b}\big\} \times \big\{C\big([0,T]; H^s\big) \cap X^2_{s,b}\big\}$$

with $(u_0, v_0) \in H^s \times H^s$. Moreover, given $t \in (0,T)$, the map $(u_0, v_0) \to (u(t), v(t))$ is Lipschitz continuous from $H^s \times H^s$ to $C((0,T); H^s) \times C((0,T); H^s)$.

Remark. A low boundedness for T (the time of existence of the solutions) can be determined by $(2C\max\{\|u_0\|_{H^s}, \|v_0\|_{H^s}\})^{-1/(b'-b)}$, where b, b' satisfy $\frac{1}{2} < b < b' < \frac{2}{3}$. This can be deduced from the proof of Theorem 1.1.

2. Preliminary estimates

The existence of unique local solution can be obtained if the estimates

$$\big\|\partial_x(u_1)u_2\bar{u}_3\big\|_{X^k_{s,b-1}} \leqslant C\|u_1\|_{X^{k_1}_{s,b}}\|u_2\|_{X^{k_2}_{s,b}}\|u_3\|_{X^{k_3}_{s,b}}, \tag{2.1}$$

$$\|u_1u_2\bar{u}_3\|_{X^k_{s,b-1}} \leqslant C\|u_1\|_{X^{k_1}_{s,b}}\|u_2\|_{X^{k_2}_{s,b}}\|u_3\|_{X^{k_3}_{s,b}} \tag{2.2}$$

hold for some $b > \frac{1}{2}$, where $u_1, u_2, u_3 = u$ or v; $k = 1, 2$, $k_m = 1, 2$ $(m = 1, 2, 3)$. In fact, we obtain the following theorem.

Theorem 2.1. *If $s \geqslant \frac{1}{4}$, $\frac{1}{2} < b < \frac{2}{3}$, $\forall b' > \frac{1}{2}$. Then*

$$\big\|\partial_x(u_1)u_2\bar{u}_3\big\|_{X^k_{s,b-1}} \leqslant C\|u_1\|_{X^{k_1}_{s,b'}}\|u_2\|_{X^{k_2}_{s,b'}}\|u_3\|_{X^{k_3}_{s,b'}}, \tag{2.3}$$

$$\|u_1u_2\bar{u}_3\|_{X^k_{s,b-1}} \leqslant C\|u_1\|_{X^{k_1}_{s,b'}}\|u_2\|_{X^{k_2}_{s,b'}}\|u_3\|_{X^{k_3}_{s,b'}}, \tag{2.4}$$

where $k = 1, 2$, $k_m = 1, 2$ $(m = 1, 2, 3)$.

Remark. Kenig et al. [8] obtained that the corresponding single trilinear estimate for the KdV equation fails for $s < \frac{1}{4}$. Therefore, our estimate seems to be best possible order.

Next, some lemmas are proven which are used in the proof of Theorem 2.1. First, we introduce the notations:

$$a = \max\left\{1, \left|\frac{2c_1}{3\varepsilon}\right|, \left|\frac{2c_2}{3\varepsilon}\right|\right\}, \qquad D_x^s = \mathcal{F}_x^{-1}|\xi|^s\mathcal{F}_x,$$

$$\mathcal{F}F_\rho(\xi,\tau) = \frac{f(\xi,\tau)}{(1+|\tau-\phi(\xi)|)^\rho} = \left\{\frac{f(\xi,\tau)}{(1+|\tau-\phi_1(\xi)|)^\rho} \text{ or } \frac{f(\xi,\tau)}{(1+|\tau-\phi_2(\xi)|)^\rho}\right\},$$

$$\|f\|_{L^p_xL^q_t} = \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|f(x,t)|^q\,dt\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}, \qquad \|f\|_{L^\infty_tH^s_x} = \big\|\|f\|_{H^s_x}\big\|_{L^\infty_t}.$$

Lemma 2.1. [12] *The groups* $\{S_j(t)\}_{-\infty}^{+\infty}$ $(j=1,2)$ *satisfy*

$$\left\| S_j(t)\varphi \right\|_{L_x^8 L_t^8} \leqslant C_j \|\varphi\|_{L^2}. \tag{2.5}$$

Lemma 2.2. *The groups* $\{S_j(t)\}_{-\infty}^{+\infty}$ $(j=1,2)$ *satisfy*

$$\left\| D_x S_j(t) P^{2a}\varphi \right\|_{L_x^\infty L_t^2} \leqslant C_j \|\varphi\|_{L^2}, \tag{2.6}$$

$$\left\| S_j(t) P^{a}\varphi \right\|_{L_x^4 L_t^\infty} \leqslant C_j \|\varphi\|_{H^{\frac{1}{4}}}, \tag{2.7}$$

$$\left\| D_x^{\frac{1}{6}} S_j(t) P^{2a}\varphi \right\|_{L_x^6 L_t^6} \leqslant C_j \|\varphi\|_{L^2}, \tag{2.8}$$

where the constant C_j *depends on* c_j *and* ε *in* (1.4).

Proof. We prove (2.6) first. It is easy to obtain that, for the same j, the two phase functions in (1.4) have the same order for the high frequency part. For simplicity, we only consider $\phi_j(\xi) = \varepsilon\xi^3 + c_j\xi^2$. As pointed out before, $\phi_j'(\xi) = 2c_j\xi + 3\varepsilon\xi^2$ has two singularities. Recalling (1.5), we know that ϕ_j is invertible if $|\xi| \geqslant 2a$. Therefore, we obtain

$$\begin{aligned} S_j(t)P^{2a}\varphi &= \int\limits_{|\xi|\geqslant 2a} e^{ix\xi} e^{it\phi_j(\xi)} \hat{\varphi}(\xi)\, d\xi = \int\limits_{|\phi_j^{-1}|\geqslant 2a} e^{ix\phi_j^{-1}} e^{it\phi_j} \hat{\varphi}\big(\phi_j^{-1}\big) \frac{1}{\phi_j'}\, d\phi_j \\ &= \mathcal{F}_t\left(e^{ix\phi_j^{-1}} \chi_{\{|\phi_j^{-1}|\geqslant 2a\}} \hat{\varphi}\big(\phi_j^{-1}\big) \frac{1}{\phi_j'} \right). \end{aligned}$$

In the following steps, using the change of variables $\xi = \phi_j^{-1}$, we have

$$\begin{aligned} \left\| S_j(t)P^{2a}\varphi \right\|_{L_t^2}^2 &= \left\| \mathcal{F}_t\left(e^{ix\phi_j^{-1}} \chi_{\{|\phi_j^{-1}|\geqslant 2a\}} \hat{\varphi}\big(\phi_j^{-1}\big) \frac{1}{\phi_j'} \right) \right\|_{L_t^2}^2 \\ &= \left\| \chi_{\{|\phi_j^{-1}|\geqslant 2a\}} \hat{\varphi}\big(\phi_j^{-1}\big) \frac{1}{\phi_j'} \right\|_{L_{\phi_j}^2}^2 = \int\limits_{|\phi_j^{-1}|\geqslant 2a} \left|\hat{\varphi}\big(\phi_j^{-1}\big)\right|^2 \frac{1}{|\phi_j'|^2}\, d\phi_j \\ &= \int\limits_{|\xi|\geqslant 2a} \left|\hat{\varphi}(\xi)\right|^2 \frac{1}{|\phi_j'(\xi)|^2} \phi_j'(\xi)\, d\xi \leqslant \int\limits_{|\xi|\geqslant 2a} \left|\hat{\varphi}(\xi)\right|^2 \frac{1}{|\phi_j'|}\, d\xi \\ &= \int\limits_{|\xi|\geqslant 2a} \frac{|\hat{\varphi}(\xi)|^2}{|3\varepsilon\xi^2|\left|1 - \frac{2c_j}{3\varepsilon\xi}\right|}\, d\xi \leqslant C_j \int\limits_{|\xi|\geqslant 2a} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|^2}\, d\xi \leqslant C_j \|\varphi\|_{\dot{H}^{-1}}^2. \end{aligned}$$

From the inequalities above, it follows that

$$\left\| S_j(t)P^{2a}\varphi \right\|_{L_t^2} \leqslant C_j \|\varphi\|_{\dot{H}^{-1}}.$$

Furthermore, we have

$$\left\| D_x S_j(t)P^{2a}\varphi \right\|_{L_x^\infty L_t^2} \leqslant C_j \|\varphi\|_{L^2}.$$

Therefore, this inequality implies the estimate (2.6) by the definition of homogeneous Sobolev space.

Let us turn to the proof of (2.7) next. One is able to recall Theorem 2.5 from [7]: *if the phase function $\phi(\xi)$ of the unitary operators $S_j(t)$ is a polynomial, then*

$$\left\|S_j(t)\varphi\right\|^2_{L^4_x L^\infty_t} \leqslant \int \left|\frac{\phi_j'(\xi)}{\phi_j''\xi)}\right|^{\frac{1}{2}} \left|\hat{\varphi}(\xi)\right|^2 d\xi.$$

Then with the help of the theorem, we can show that

$$\begin{aligned}\left\|S_j(t)P^a\varphi\right\|^2_{L^4_x L^\infty_t} &\leqslant \int \left|\mathcal{F}P^a\varphi(\xi)\right|^2 \left|\frac{\phi_j'(\xi)}{\phi_j''(\xi)}\right|^{\frac{1}{2}} d\xi \leqslant \int \left|\mathcal{F}P^a\varphi(\xi)\right|^2 \left|\frac{3\varepsilon\xi^2+2c_j\xi}{6\varepsilon\xi+2c_j}\right|^{\frac{1}{2}} d\xi \\ &\leqslant \int \left|\mathcal{F}P^a\varphi(\xi)\right|^2 \left(\frac{|3\varepsilon\xi^2|\left|1-a\frac{1}{\xi}\right|}{|6\varepsilon\xi|\left|1-\frac{1}{2}a\frac{1}{\xi}\right|}\right)^{\frac{1}{2}} d\xi \\ &\leqslant \int \left|\mathcal{F}P^a\varphi(\xi)\right|^2 \left(\frac{|3\varepsilon\xi^2|\left(1+a\frac{1}{a}\right)}{|6\varepsilon\xi|\left(1-\frac{1}{2}a\frac{1}{a}\right)}\right)^{\frac{1}{2}} d\xi \leqslant C_j\left\|P^a\varphi\right\|^2_{\dot{H}^{\frac{1}{4}}}.\end{aligned}$$

Therefore, it follows the estimate (2.7).

Finally, we will use Stein's interpolation theorem [9,10] listed as follows: *suppose* $\{T_z\}$, $0 \leqslant \operatorname{Re} z \leqslant 1$, *is an admissible family of linear operators satisfying*

$$\|T_{iy}f\|_{L^{q_0}} \leqslant M_0(y)\|f\|_{L^{p_0}},$$
$$\|T_{1+iy}f\|_{L^{q_1}} \leqslant M_1(y)\|f\|_{L^{p_1}}$$

for all simple functions f in $L^1(M,m,\mu)$, where $1 \leqslant p_j, q_j \leqslant \infty$, $M_j(y)$, $j=0,1$, are independent of f and satisfy

$$\sup_{-\infty<y<\infty} e^{-b|y|}\ln M_j(y) < \infty$$

for some $b<\pi$. Then, if $0 \leqslant t \leqslant 1$, there exists a constant M_t, such that

$$\|T_t f\|_{L^{q_t}} \leqslant M_t\|f\|_{L^{p_t}}$$

for all simple functions f provided

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \text{and} \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Therefore, we consider the analytic family of operators $T_z P^{2a}\varphi = D_x^{-z/4} D_x^{1-z} S(t) P^{2a}\varphi$, with $z \in \mathbb{C}$, $0 \leqslant \operatorname{Re} z \leqslant 1$, where $z = iy$,

$$T_{iy}P^{2a}\varphi = \frac{\partial}{\partial x} S(t) D_x^{-i5y/4}\mathcal{H}P^{2a}\varphi,$$

where $\mathcal{H}$ denotes the Hilbert transform. Thus by (2.6),

$$\left\|T_{iy}P^{2a}\varphi\right\|_{L^\infty_x L^2_t} \leqslant C\|\varphi\|_{L^2}.$$

Since $\|D_x^{iy}\mathcal{H}\varphi\|_{L^2} = \|\varphi\|_{L^2}$. When $z = 1+iy$, $T_{1+iy}P^{2a}\varphi = D_x^{-1/4} D_x^{-i5y/4} S(t)P^{2a}\varphi$, and by (2.7),

$$\left\|T_{1+iy}P^{2a}\varphi\right\|_{L^4_x L^\infty_t} \leqslant C\|\varphi\|_{L^2}.$$

Hence, using the above Stein's interpolation theorem, we obtain the result (2.8) by choosing $z = \frac{2}{3}$. □

Lemma 2.3. [6] *If $\rho > \frac{1}{2}$, for any fixed N with $0 < N < +\infty$, it holds that*

$$\|P_N F_\rho\|_{L_x^2 L_t^\infty} \leqslant C\|f\|_{L_\xi^2 L_\tau^2}, \tag{2.9}$$

where the constant C does depend on N.

Lemma 2.4. *If $\rho > \frac{1}{2}\frac{4(q-2)}{3q}$ for $2 \leqslant q \leqslant 8$. Then*

$$\|F_\rho\|_{L_x^q L_t^q} \leqslant C_j\|f\|_{L_\xi^2 L_\tau^2}, \tag{2.10}$$

where the constants C_j depends on ε and c_j.

Proof. Changing variables to $\tau = \lambda + \phi_j(\xi)$ where $\phi_j(\xi)$ can be taken any form in (1.4), we have

$$\begin{aligned} F_\rho(x,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(x\xi+t\tau)}\frac{f(\xi,\tau)}{(1+|\tau-\phi_j(\xi)|)^\rho}\,d\xi\,d\tau \\ &= \int_{-\infty}^{\infty} e^{it\lambda}\left(\int_{-\infty}^{\infty} e^{i(x\xi+t\phi_j(\xi))} f\big(\xi,\lambda+\phi_j(\xi)\big)\,d\xi\right)\frac{d\lambda}{(1+|\lambda|)^\rho}. \end{aligned}$$

Therefore, using (2.5), Minkowski's integral inequality and taking $\rho > \frac{1}{2}$, we can prove that

$$\|F_\rho\|_{L_x^8 L_t^8} \leqslant C_j \int_{-\infty}^{+\infty} \big\|f\big(\xi,\lambda+\phi_j(\xi)\big)\big\|_{L_\xi^2}\frac{d\lambda}{(1+|\lambda|)^\rho} \leqslant C_j\|f\|_{L_\xi^2 L_\tau^2}. \tag{2.11}$$

By interpolation between (2.11) and

$$\|F_0\|_{L_x^2 L_t^2} = \|f\|_{L_\xi^2 L_\tau^2}, \tag{2.12}$$

we obtain, for $\rho > \frac{1}{2}\frac{4(q-2)}{3q}$, that

$$\|F_\rho\|_{L_x^q L_t^q} \leqslant C_j\|f\|_{L_\xi^2 L_\tau^2}. \tag{2.13}$$

In fact, we consider the analytic family of operators

$$T_z f = \mathcal{F}^{-1}\frac{f}{\langle\sigma\rangle^{zb}}, \quad z \in \mathbb{C},\ 0 \leqslant \operatorname{Re} z \leqslant 1.$$

If $z = 1 + iy$, then

$$T_{1+iy} f = \mathcal{F}^{-1}\frac{f}{\langle\sigma\rangle^{(1+iy)b}}.$$

Thus, (2.11) gives

$$\|T_{1+iy} f\|_{L_x^8 L_t^8} \leqslant C\|f\|_{L_\xi^2 L_\tau^2}.$$

If $z = iy$, then

$$T_{iy} f = \mathcal{F}^{-1}\frac{f}{\langle\sigma\rangle^{(iy)b}}.$$

Thus, (2.12) gives

$$\|T_{iy} f\|_{L_x^2 L_t^2} \leqslant C\|f\|_{L_\xi^2 L_\tau^2}.$$

Therefore, by Stein's analytic interpolation theorem, we can obtain the result (2.13) with $z = \frac{4(q-2)}{3q}$. □

Lemma 2.5. *Let $\rho > \frac{\theta}{2}$ with $\theta \in [0,1]$. Then*

$$\left\| D_x^\theta P^{2a} F_\rho \right\|_{L_x^{\frac{2}{1-\theta}} L_t^2} \leqslant C_j \|f\|_{L_\xi^2 L_\tau^2}, \tag{2.14}$$

$$\left\| D_x^{-\frac{1}{4}} P^{2a} F_\rho \right\|_{L_x^4 L_t^\infty} \leqslant C_j \|f\|_{L_\xi^2 L_\tau^2}, \tag{2.15}$$

where the constants C_j does depend on ε and c_j.

Proof. From the argument (2.11) and inequality (2.6), we have for $\rho > \frac{1}{2}$,

$$\left\| D_x P^{2a} F_\rho \right\|_{L_x^\infty L_t^2} \leqslant C_j \|f\|_{L_\xi^2 L_\tau^2}.$$

Then (2.14) follows by Stein's interpolation between the inequality above and (2.12). Similarly to the above argument, we obtain (2.15) from inequality (2.7). □

Lemma 2.6. *If f, f_1, f_2 and f_3 belong to Schwartz space on $\mathbb{R}^2$, then*

$$\int_\star \bar{\hat{f}}(\xi,\tau)\hat{f}_1(\xi_1,\tau_1)\hat{f}_2(\xi_2,\tau_2)\hat{f}_3(\xi_3,\tau_3)\,d\delta = \int \bar{f} f_1 f_2 f_3(x,t)\,dx\,dt. \tag{2.16}$$

Proof. For simplicity, we only discuss the case of one variable. In fact, it follows that

$$\begin{aligned}
&\int_{\xi=\xi_1+\xi_2+\xi_3} \bar{\hat{f}}(\xi)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)\,d\delta \\
&\quad = \int_{\xi=\xi_1+\xi_2+\xi_3} \hat{\bar{f}}(-\xi)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)\,d\delta \\
&\quad = \int_{\xi_1}\int_{\xi_2'}\int_{\xi_3'} \hat{\bar{f}}\left(-\xi_3'\right)\hat{f}_1(\xi_1)\hat{f}_2\left(\xi_2'-\xi_1\right)\hat{f}_3\left(\xi_3'-\xi_2'\right)d\xi_1\,d\xi_2'\,d\xi_3' \\
&\quad = \hat{\bar{f}} * \hat{f}_1 * \hat{f}_2 * \hat{f}_3(0) = \mathcal{F}\bar{f} f_1 f_2 f_3(0) \\
&\quad = \int \bar{f} f_1 f_2 f_3(x)\,dx. \qquad \square
\end{aligned}$$

Lemma 2.7. [6,8] *Let $s \in \mathbb{R}$, $\frac{1}{2} < b < b' < 1$, $0 < \delta < 1$. Then it holds that*

$$\left\|\psi_\delta(t)S(t)u_0\right\|_{X_{s,b}} \leqslant C\delta^{\frac{1}{2}-b}\|u_0\|_{H^s}, \tag{2.17}$$

$$\left\|\psi_\delta(t)\int_0^t S(t-t')f(t')\,dt'\right\|_{X_{s,b}} \leqslant C\delta^{\frac{1}{2}-b}\|f\|_{X_{s,b-1}}, \tag{2.18}$$

$$\left\|\psi_\delta(t)\int_0^t S(t-t')f(t')\,dt'\right\|_{H^s} \leqslant C\delta^{\frac{1}{2}-b}\|f\|_{X_{s,b-1}}, \tag{2.19}$$

$$\left\|\psi_\delta(t)F\right\|_{X_{s,b}} \leqslant C\delta^{\frac{1}{2}-b}\|F\|_{X_{s,b}}, \tag{2.20}$$

$$\left\|\psi_\delta(t)F\right\|_{X_{s,b-1}} \leqslant C\delta^{b'-b}\|F\|_{X_{s,b'-1}}. \tag{2.21}$$

We give the proof of Theorem 2.1 now. We will prove (2.3) only, the proof of (2.4) is similar to that of (2.3), we omit the details here. By duality and the Plancheral identity, it suffices to show that

$$\begin{aligned}
\Upsilon &= \int_\star \langle\xi\rangle^s i\xi_1 \frac{\bar f(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\mathcal{F}u_1(\tau_1,\xi_1)\mathcal{F}u_2(\tau_2,\xi_2)\mathcal{F}\bar u_3(\tau_3,\xi_3)\,d\delta \\
&= \int_\star \frac{\langle\xi\rangle^s i\xi_1}{\langle\sigma\rangle^{1-b}\prod_{k=1}^3\langle\xi_k\rangle^s\langle\sigma_k\rangle^{b'}}\bar f(\tau,\xi)f_1(\tau_1,\xi_1)f_2(\tau_2,\xi_2)f_3(\tau_3,\xi_3)\,d\delta \\
&\leqslant C\|f\|_{L^2}\prod_{k=1}^3\|f_k\|_{L^2},
\end{aligned}$$

for all $\bar f\in L^2$, $\bar f\geqslant 0$, where

$$f_k = \langle\xi_k\rangle^s\langle\sigma_k\rangle^{b'}\hat u_k, \quad k=1,2; \qquad f_3=\langle\xi_3\rangle^s\langle\sigma_3\rangle^{b'}\hat{\bar u}_3,$$
$$\xi=\xi_1+\xi_2+\xi_3, \qquad \tau=\tau_1+\tau_2+\tau_3.$$

Without loss of generality, we can assume that $f_k\geqslant 0$ for $k=1,2,3$.

$$\mathcal{F}F_\rho^k(\xi,\tau) = \left\{\frac{f_k(\xi,\tau)}{(1+|\tau-\varepsilon\xi^3-c_1\xi^2|)^\rho} \text{ or } \frac{f_k(\xi,\tau)}{(1+|\tau-\varepsilon\xi^3-c_2\xi^2|)^\rho}\right\} = \frac{f_k}{\langle\sigma_k\rangle^\rho},$$

where $k=1,2$.

$$\mathcal{F}F_\rho^3(\xi,\tau) = \left\{\frac{f_3(\xi,\tau)}{(1+|\tau-\varepsilon\xi^3+c_1\xi^2|)^\rho} \text{ or } \frac{f_3(\xi,\tau)}{(1+|\tau-\varepsilon\xi^3+c_2\xi^2|)^\rho}\right\} = \frac{f_3}{\langle\sigma_3\rangle^\rho}.$$

In order to obtain the boundedness of the integral Υ, we split the domain of integration in several pieces.

Situation I. Assume: $|\xi|\leqslant 6a$.

Case 1. If $|\xi_1|\leqslant 2a$. Then, by Lemmas 2.4 and 2.6, the integral Υ restricted to this domain is bounded by

$$\begin{aligned}
&\int_\star \frac{\langle\xi\rangle^s\bar f(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\frac{|\xi_1|f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}\langle\xi_1\rangle^s}\frac{f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}\langle\xi_2\rangle^s}\frac{f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}\langle\xi_3\rangle^s}\,d\delta \\
&\quad\leqslant C\int_\star \frac{\bar f(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\frac{f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}}\frac{f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}}\frac{f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}}\,d\delta \\
&\quad\leqslant C\int \overline{F}_{1-b}\cdot F_{b'}^1\cdot F_{b'}^2\cdot F_{b'}^3(x,t)\,dx\,dt
\end{aligned}$$

$$\leqslant C\|F_{1-b}\|_{L_x^2L_t^2}\|F_{b'}^1\|_{L_x^6L_t^6}\|F_{b'}^2\|_{L_x^6L_t^6}\|F_{b'}^3\|_{L_x^6L_t^6}$$
$$\leqslant C\|f\|_{L_\xi^2L_\tau^2}\|f_1\|_{L_\xi^2L_\tau^2}\|f_2\|_{L_\xi^2L_\tau^2}\|f_3\|_{L_\xi^2L_\tau^2}.$$

Case 2. Assume $|\xi_1|\geqslant 2a$.

Subcase (1). If $|\xi_2|\leqslant 2a$ or $|\xi_3|\leqslant 2a$ (without loss of generality, we can assume $|\xi_2|\leqslant 2a$), then by Lemmas 2.3–2.6, the integral Υ restricted to this domain is bounded by

$$\int_\star \frac{\langle\xi\rangle^s\bar f(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\frac{|\xi_1|\chi_{|\xi_1|\geqslant 2a}f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}\langle\xi_1\rangle^s}\frac{\chi_{|\xi_2|\leqslant 2a}f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}\langle\xi_2\rangle^s}\frac{f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}\langle\xi_3\rangle^s}\,d\delta$$
$$\leqslant C\int_\star \frac{\bar f(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\frac{|\xi_1|\chi_{|\xi_1|\geqslant 2a}f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}}\frac{\chi_{|\xi_2|\leqslant 2a}f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}}\frac{f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}}\,d\delta$$
$$\leqslant C\int \overline{F}_{1-b}\cdot D_xP^{2a}F_{b'}^1\cdot P_{2a}F_{b'}^2\cdot F_{b'}^3(x,t)\,dx\,dt$$
$$\leqslant C\|F_{1-b}\|_{L_x^4L_t^4}\|D_xP^{2a}F_{b'}^1\|_{L_x^\infty L_t^2}\|P_{2a}F_{b'}^2\|_{L_x^2L_t^\infty}\|F_{b'}^3\|_{L_x^4L_t^4}$$
$$\leqslant C\|f\|_{L_\xi^2L_\tau^2}\|f_1\|_{L_\xi^2L_\tau^2}\|f_2\|_{L_\xi^2L_\tau^2}\|f_3\|_{L_\xi^2L_\tau^2}.$$

Subcase (2). If $|\xi_2|\geqslant 2a$ and $|\xi_3|\geqslant 2a$, then by Lemmas 2.3–2.6, for $s\geqslant\frac14$, the integral Υ is bounded by

$$\int_\star \frac{\langle\xi\rangle^s\bar f(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\frac{|\xi_1|\chi_{|\xi_1|\geqslant 2a}f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}\langle\xi_1\rangle^s}\frac{\chi_{|\xi_2|\geqslant 2a}f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}\langle\xi_2\rangle^s}\frac{\chi_{|\xi_3|\geqslant 2a}f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}\langle\xi_3\rangle^s}\,d\delta$$
$$\leqslant C\int_\star \frac{\bar f(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\frac{|\xi_1|\chi_{|\xi_1|\geqslant 2a}f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}}\frac{\chi_{|\xi_2|\geqslant 2a}f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}\langle\xi_2\rangle^s}\frac{\chi_{|\xi_3|\geqslant 2a}f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}\langle\xi_3\rangle^s}\,d\delta$$
$$\leqslant C\int \overline{F}_{1-b}\cdot D_xP^{2a}F_{b'}^1\cdot P^{2a}D_x^{-\frac14}F_{b'}^2\cdot P^{2a}D_x^{-\frac14}F_{b'}^3(x,t)\,dx\,dt$$
$$\leqslant C\|F_{1-b}\|_{L_x^2L_t^2}\|D_xP^{2a}F_{b'}^1\|_{L_x^\infty L_t^2}\|P^{2a}D_x^{-\frac14}F_{b'}^2\|_{L_x^4L_t^\infty}\|P^{2a}D_x^{-\frac14}F_{b'}^3\|_{L_x^4L_t^\infty}$$
$$\leqslant C\|f\|_{L_\xi^2L_\tau^2}\|f_1\|_{L_\xi^2L_\tau^2}\|f_2\|_{L_\xi^2L_\tau^2}\|f_3\|_{L_\xi^2L_\tau^2}.$$

Situation II. Assume: $|\xi|\geqslant 6a$.

Case 1. If $|\xi_1|\leqslant 2a$, then $|\xi|\leqslant 3\max(|\xi_2|,|\xi_3|)$ (without loss of generality, we can assume $2a\leqslant\frac13|\xi|\leqslant|\xi_3|$). Then by Lemmas 2.4 and 2.6, the integral Υ is bounded by

$$\int_\star \frac{\langle\xi\rangle^s\bar f(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\frac{|\xi_1|\chi_{|\xi_1|\leqslant 2a}f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}\langle\xi_1\rangle^s}\frac{f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}\langle\xi_2\rangle^s}\frac{\chi_{|\xi_3|\geqslant 2a}f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}\langle\xi_3\rangle^s}\,d\delta$$
$$\leqslant C\int_\star \frac{\bar f(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\frac{\chi_{|\xi_1|\leqslant 2a}f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}}\frac{f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}}\frac{\chi_{|\xi_3|\geqslant 2a}f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}}\,d\delta$$
$$\leqslant C\int \overline{F}_{1-b}\cdot P_{2a}F_{b'}^1\cdot F_{b'}^2\cdot P^{2a}F_{b'}^3(x,t)\,dx\,dt$$

$$\leqslant C\|F_{1-b}\|_{L_x^2L_t^2}\|P_{2a}F_{b'}^1\|_{L_x^6L_t^6}\|F_{b'}^2\|_{L_x^6L_t^6}\|P^{2a}F_{b'}^3\|_{L_x^6L_t^6}$$
$$\leqslant C\|f\|_{L_\xi^2L_\tau^2}\|f_1\|_{L_\xi^2L_\tau^2}\|f_2\|_{L_\xi^2L_\tau^2}\|f_3\|_{L_\xi^2L_\tau^2}.$$

Case 2. Assume $|\xi_1|\geqslant 2a$, we distinguish the different situations.

Subcase (1). If $|\xi_2|\leqslant 2a$ or $|\xi_3|\leqslant 2a$ (without loss of generality, we can assume $|\xi_3|\leqslant 2a$), then $|\xi|\leqslant 3\max(|\xi_1|,|\xi_2|)$. Then by Lemmas 2.3–2.6, the integral Υ restricted to this domain is bounded by

$$\int\limits_\star \frac{\langle\xi\rangle^s\bar f(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\frac{|\xi_1|\chi_{|\xi_1|\geqslant 2a}f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}\langle\xi_1\rangle^s}\frac{f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}\langle\xi_2\rangle^s}\frac{\chi_{|\xi_3|\leqslant 2a}f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}\langle\xi_3\rangle^s}\,d\delta$$
$$\leqslant C\int\limits_\star \frac{\bar f(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\frac{|\xi_1|\chi_{|\xi_1|\geqslant 2a}f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}}\frac{f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}}\frac{\chi_{|\xi_3|\leqslant 2a}f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}}\,d\delta$$
$$\leqslant C\int \overline{F}_{1-b}\cdot D_xP^{2a}F_{b'}^1\cdot F_{b'}^2\cdot P_{2a}F_{b'}^3(x,t)\,dx\,dt$$
$$\leqslant C\|F_{1-b}\|_{L_x^4L_t^4}\|D_xP^{2a}F_{b'}^1\|_{L_x^\infty L_t^2}\|F_{b'}^2\|_{L_x^4L_t^4}\|P_{2a}F_{b'}^3\|_{L_x^2L_t^\infty}$$
$$\leqslant C\|f\|_{L_\xi^2L_\tau^2}\|f_1\|_{L_\xi^2L_\tau^2}\|f_2\|_{L_\xi^2L_\tau^2}\|f_3\|_{L_\xi^2L_\tau^2}.$$

Subcase (2). Assume $|\xi_2|\geqslant 2a$ and $|\xi_3|\geqslant 2a$. We have to discuss two situations separately.

(i) If $|\xi|\leqslant 3\max(|\xi_1|,|\xi_2|,|\xi_3|)=3|\xi_1|$, then by Lemmas 2.3–2.6, for $s\geqslant\frac14$, the integral Υ is bounded by

$$\int\limits_\star \frac{\chi_{|\xi|\geqslant 6a}\langle\xi\rangle^s f(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\frac{\chi_{|\xi_1|\geqslant 2a}|\xi_1|f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}\langle\xi_1\rangle^s}\frac{\chi_{|\xi_2|\geqslant 2a}f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}\langle\xi_2\rangle^s}\frac{\chi_{|\xi_3|\geqslant 2a}f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}\langle\xi_3\rangle^s}\,d\delta$$
$$\leqslant C\int P^{6a}F_{1-b}\cdot D_xP^{2a}F_{b'}^1\cdot D_x^{-s}P^{2a}F_{b'}^2\cdot D_x^{-s}P^{2a}F_{b'}^3(x,t)\,dx\,dt$$
$$\leqslant C\|P^{6a}F_{1-b}\|_{L_x^2L_t^2}\|D_xP^{2a}F_{b'}^1\|_{L_x^\infty L_t^2}\|D_x^{-s}P^{2a}F_{b'}^2\|_{L_x^4L_t^\infty}\|D_x^{-s}P^{2a}F_{b'}^3\|_{L_x^4L_t^\infty}$$
$$\leqslant C\|f\|_{L_\xi^2L_\tau^2}\|f_1\|_{L_\xi^2L_\tau^2}\|f_2\|_{L_\xi^2L_\tau^2}\|f_3\|_{L_\xi^2L_\tau^2}.$$

(ii) If $|\xi|\leqslant 3\max(|\xi_1|,|\xi_2|,|\xi_3|)=3|\xi_2|$, then by symmetry, we have three cases as follows. If $s\geqslant 1$, then by Lemmas 2.4 and 2.6, the integral Υ restricted to this domain is bounded by

$$\int\limits_\star \frac{|\xi_1|\langle\xi\rangle^s f(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\frac{f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}\langle\xi_1\rangle^s}\frac{f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}\langle\xi_2\rangle^s}\frac{f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}\langle\xi_3\rangle^s}\,d\delta$$
$$\leqslant C\int\limits_\star \frac{f(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\frac{f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}}\frac{f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}}\frac{f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}}\,d\delta$$
$$\leqslant C\int F_{1-b}\cdot F_{b'}^1\cdot F_{b'}^2\cdot F_{b'}^3(x,t)\,dx\,dt$$
$$\leqslant C\|F_{1-b}\|_{L_x^4L_t^4}\|F_{b'}^1\|_{L_x^4L_t^4}\|F_{b'}^2\|_{L_x^4L_t^4}\|F_{b'}^3\|_{L_x^4L_t^4}$$
$$\leqslant C\|f\|_{L_\xi^2L_\tau^2}\|f_1\|_{L_\xi^2L_\tau^2}\|f_2\|_{L_\xi^2L_\tau^2}\|f_3\|_{L_\xi^2L_\tau^2}.$$

If $\frac12\leqslant s\leqslant 1$, then by Lemmas 2.4–2.6, the integral Υ restricted to this domain is bounded by

$$\int_\star \frac{\langle\xi\rangle^s f(\tau,\xi)}{\langle\sigma\rangle^{1-b}} \frac{|\xi_1|\chi_{|\xi_1|\geqslant 2a} f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}\langle\xi_1\rangle^s} \frac{f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}\langle\xi_2\rangle^s} \frac{\chi_{|\xi_3|\geqslant 2a} f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}\langle\xi_3\rangle^s}\,d\delta$$

$$\leqslant C\int_\star \frac{f(\tau,\xi)}{\langle\sigma\rangle^{1-b}} \frac{\chi_{|\xi_1|\geqslant 2a}|\xi_1|^{1-s} f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}} \frac{f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}} \frac{\chi_{|\xi_3|\geqslant 2a} f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}\langle\xi_3\rangle^s}\,d\delta$$

$$\leqslant C\int F_{1-b}\cdot D_x^{\frac12}P^{2a}F_{b'}^1\cdot F_{b'}^2\cdot D_x^{-\frac14}P^{2a}F_{b'}^3(x,t)\,dx\,dt$$

$$\leqslant C\|F_{1-b}\|_{L_x^4L_t^4}\big\|D_x^{\frac12}P^{2a}F_{b'}^1\big\|_{L_x^4L_t^2}\big\|F_{b'}^2\big\|_{L_x^4L_t^4}\big\|D_x^{-\frac14}P^{2a}F_{b'}^3\big\|_{L_x^4L_t^\infty}$$

$$\leqslant C\|f\|_{L_\xi^2L_\tau^2}\|f_1\|_{L_\xi^2L_\tau^2}\|f_2\|_{L_\xi^2L_\tau^2}\|f_3\|_{L_\xi^2L_\tau^2}.$$

If $\frac14\leqslant s\leqslant\frac12$, then by Lemmas 2.4–2.6, for $1-b\geqslant\frac{s}{2}$, the integral Υ is bounded by

$$\int_\star \frac{\langle\xi\rangle^s\chi_{|\xi|\geqslant 6a} f(\tau,\xi)}{\langle\sigma\rangle^{1-b}} \frac{|\xi_1|\chi_{|\xi_1|\geqslant 2a} f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}\langle\xi_1\rangle^s} \frac{\chi_{|\xi_2|\geqslant 2a} f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}\langle\xi_2\rangle^s} \frac{\chi_{|\xi_3|\geqslant 2a} f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}\langle\xi_3\rangle^s}\,d\delta$$

$$\leqslant C\int_\star \frac{|\xi|^s\chi_{|\xi|\geqslant 6a} f(\tau,\xi)}{\langle\sigma\rangle^{1-b}} \frac{\chi_{|\xi_1|\geqslant 2a}|\xi_1|^{1-s} f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}} \frac{\chi_{|\xi_2|\geqslant 2a} f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}\langle\xi_2\rangle^s}$$

$$\times\frac{\chi_{|\xi_3|\geqslant 2a} f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}\langle\xi_3\rangle^s}\,d\delta$$

$$\leqslant C\int D_x^sP^{6a}F_{1-b}\cdot D_x^{1-s}P^{2a}F_{b'}^1\cdot D_x^{-s}P^{2a}F_{b'}^2\cdot D_x^{-s}P^{2a}F_{b'}^3(x,t)\,dx\,dt$$

$$\leqslant C\big\|D_x^sP^{6a}F_{1-b}\big\|_{L_x^{\frac{2}{1-s}}L_t^2}\big\|D_x^{1-s}P^{2a}F_{b'}^1\big\|_{L_x^{\frac{2}{s}}L_t^2}\big\|D_x^{-\frac14}P^{2a}F_{b'}^2\big\|_{L_x^4L_t^\infty}\big\|D_x^{-\frac14}P^{2a}F_{b'}^3\big\|_{L_x^4L_t^\infty}$$

$$\leqslant C\|f\|_{L_\xi^2L_\tau^2}\|f_1\|_{L_\xi^2L_\tau^2}\|f_2\|_{L_\xi^2L_\tau^2}\|f_3\|_{L_\xi^2L_\tau^2}.$$

This completes the proof of Theorem 2.1. □

3. The proof of Theorem 1.1

For $(u_0,v_0)\in H^s\times H^s$ $(s\geqslant 1/4)$, define the operator as follows:

$$\Phi_{u_0}(u)=\Phi(u)=\psi_1(t)S_1(t)u_0-\psi_1(t)i\int_0^t S_1(t-t')\psi_\delta(t')\big[2\big(\alpha|u|^2+\beta|v|^2\big)u$$
$$-i\varepsilon\big\{\big(2\mu_1|u|^2+\nu_1|v|^2\big)u_x+\nu_1u\bar v v_x\big\}\big]\,dt',$$

$$\Psi_{v_0}(u)=\Psi(u)=\psi_1(t)S_2(t)v_0-\psi_1(t)i\int_0^t S_2(t-t')\psi_\delta(t')\big[2\big(\beta|u|^2+\gamma|v|^2\big)v$$
$$-i\varepsilon\big\{\big(\nu_2|u|^2+2\mu_2|v|^2\big)v_x+\nu_2\bar u v u_x\big\}\big]\,dt',$$

and the sets

$$\mathcal{B}=\big\{u\in X_{s,b}^1\colon\ \|u\|_{X_{s,b}^1}\leqslant 2C\|u_0\|_{H^s}\big\},$$
$$\mathcal{C}=\big\{v\in X_{s,b}^2\colon\ \|v\|_{X_{s,b}^2}\leqslant 2C\|v_0\|_{H^s}\big\}.$$

In order to show that (Φ, Ψ) is a contraction on $\mathcal{B} \times \mathcal{C}$, we first prove that

$$(\Phi, \Psi)(\mathcal{B} \times \mathcal{C}) \subset \mathcal{B} \times \mathcal{C}.$$

By Theorem 2.1 and Lemma 2.7, we obtain, for $b < b' < \frac{2}{3}$, the following chain of inequalities:

$$\begin{aligned}
\left\|\Phi(u)\right\|_{X^1_{s,b}} &\leqslant C\|u_0\|_{H^s} + C\delta^{b'-b}\bigl\{2\bigl(\alpha\|u\|^2_{X^1_{s,b}} + \beta\|v\|^2_{X^2_{s,b}}\bigr)\|u\|_{X^1_{s,b}}\bigr\} \\
&\quad + \varepsilon\bigl(2\mu_1\|u\|^2_{X^1_{s,b}} + \nu_1\|v\|^2_{X^2_{s,b}}\bigr)\|u\|_{X^1_{s,b}} + \varepsilon\nu_1\|u\|_{X^1_{s,b}}\|v\|^2_{X^2_{s,b}} \\
&\leqslant C\|u_0\|_{H^s} + C\delta^{b'-b}\max\bigl(\|u_0\|_{H^s}, \|v_0\|_{H^s}\bigr)^2\|u\|_{X^1_{s,b}}, \\
\left\|\Psi(v)\right\|_{X^2_{s,b}} &\leqslant C\|v_0\|_{H^s} + C\delta^{b'-b}\bigl\{2\bigl(\gamma\|v\|^2_{X^2_{s,b}} + \beta\|u\|^2_{X^1_{s,b}}\bigr)\|v\|_{X^2_{s,b}}\bigr\} \\
&\quad + \varepsilon\bigl(2\mu_2\|v\|^2_{X^2_{s,b}} + \nu_2\|u\|^2_{X^1_{s,b}}\bigr)\|v\|_{X^2_{s,b}} + \varepsilon\nu_2\|v\|_{X^2_{s,b}}\|u\|^2_{X^1_{s,b}} \\
&\leqslant C\|v_0\|_{H^s} + C\delta^{b'-b}\max\bigl(\|u_0\|_{H^s}, \|v_0\|_{H^s}\bigr)^2\|v\|_{X^2_{s,b}}.
\end{aligned}$$

If we fix δ such that $C\delta^{b'-b}\max(\|u_0\|^2_{H^s}, \|v_0\|^2_{H^s}) \leqslant \frac{1}{2}$, then

$$(\Phi, \Psi)(\mathcal{B} \times \mathcal{C}) \subset (\mathcal{B} \times \mathcal{C}).$$

For $(u_j, v_j) \in \mathcal{B} \times \mathcal{C}$, $j = 1, 2$. In an analogous way to the above, we obtain that

$$\begin{aligned}
\left\|\Phi(u_1) - \Phi(u_2)\right\|_{X^1_{s,b}} &\leqslant \frac{1}{2}\|u_1 - u_2\|_{X^1_{s,b}}, \\
\left\|\Psi(v_1) - \Psi(v_2)\right\|_{X^2_{s,b}} &\leqslant \frac{1}{2}\|v_1 - v_2\|_{X^2_{s,b}}.
\end{aligned}$$

Therefore, (Φ, Ψ) is a contraction map on $\mathcal{B} \times \mathcal{C}$. Thus there exists a unique fixed point which solves the Cauchy problem (1.1) for $T < \delta$.

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